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GEOMETRY: EXAMPLES 3

1. Let $\Sigma \subset \mathbb{R}^3$ be a surface of revolution parametrised by
$$\sigma(u, v) = (f(u)\cos v, f(u)\sin v, g(u))$$
as usual. Let $\gamma : I \to \Sigma$ be a geodesic, where I is an open interval, and let $\psi(t)$ be a smooth choice of angle from $\dot{\gamma}(t)$ to the parallel through $\gamma(t)$. Prove *Clairaut's relation*: that $f(\gamma(t))\cos\psi(t)$ is constant. If you stand on the equator, facing directly along it, then turn through angle α towards the north and walk straight ahead, what is the maximum latitude you will reach?

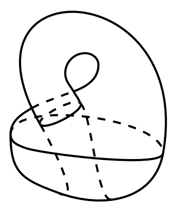
2. Show that if $H : \Sigma_1 \to \Sigma_2$ is a local isometry between embedded surfaces then a curve $\gamma : I \to \Sigma_1$ is a geodesic in Σ_1 iff $F \circ \gamma$ is a geodesic in Σ_2.

3. Fix $a > 0$ and let Σ be the open half-cone $\{(x, y, z) : z^2 = a(x^2 + y^2), z > 0\}$. Let S denote the slit plane $\mathbb{R}^2 \setminus \{(x, 0) : x \leq 0\}$.
   (a) By rolling up S into a cone, construct an explicit local isometry $H : S \to \Sigma$.
   (b) Hence show that a geodesic on Σ is complete if and only if it's not contained in a meridian.
   (c) Show also that for $a \leq 3$ no geodesic on Σ intersects itself, but that for $a > 3$ every complete geodesic intersects itself.
   *(d) How many times does a complete geodesic intersect itself?

4. Fix an embedded surface Σ and a smooth path $\gamma : [t_0, t_1] \to \Sigma$. Show that if $\Gamma : (-\varepsilon, \varepsilon) \times [t_0, t_1] \to \Sigma$ is a smooth map with $\Gamma(0, t) = \gamma(t)$ for all t then the map $\mathcal{E} : (-\varepsilon, \varepsilon) \to \mathbb{R}$ given by
$$\mathcal{E}(s) = \mathrm{Energy}(\Gamma(s, -))$$
is differentiable at $s = 0$ with
$$\mathcal{E}'(0) = 2\int_{t_0}^{t_1} \dot{\gamma}(t) \cdot \Gamma_{st}(0, t)\,\mathrm{d}t.$$
[*Hint: Taylor expand Γ_t in the s-direction and use compactness of $[-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}] \times [t_0, t_1]$ to bound the error.*]

5. Let X be a topological space that's locally homeomorphic to $\mathbb{R}^2$. Show that:
   (a) X is connected iff it's path-connected.
   (b) X is second-countable iff it's *Lindelöf* (every open cover has a countable subcover) iff it can be covered by countably many charts.
   *(c) X is Hausdorff iff it's *regular* (given a closed set C and a point $p \in X \setminus C$ there exist disjoint open sets in X containing p and C respectively).

6. (a) Viewing T^2 as $\mathbb{R}^2/\mathbb{Z}^2$, show that the map $H : \mathbb{R}^2 \to \mathbb{R}^2$ given by $(x, y) \mapsto (x + \frac{1}{2}, -y)$ induces a diffeomorphism $h : T^2 \to T^2$. [*Hint: The quotient map $Q : \mathbb{R}^2 \to T^2$ is a local diffeomorphism.*]
   (b) Show moreover that h generates an action of $\mathbb{Z}/2$ on T^2 which is free and proper, and deduce that the quotient K is an abstract smooth surface. This is the *Klein bottle*.
   (c) Draw a fundamental domain for K and indicate the edge identifications. Hence show that K contains an open subset diffeomorphic to a Möbius band, and deduce that K is non-orientable.
   (d) Convince yourself that the diagram shows the image of a smooth map $i : K \to \mathbb{R}^3$. On your fundamental domain for a K, draw the set S of points where i fails to be injective. On a fundamental domain for T^2 draw the preimage of S under the quotient map $q : T^2 \to K$.



7. Equip the open unit disc with the Riemannian metric

$$\frac{\mathrm{d}u^2 + \mathrm{d}v^2}{1 - u^2 - v^2}.$$

Prove directly that diameters are length-minimizing curves. Show that distances in the metric are bounded, but that areas can be unbounded.

8. Consider $\mathbb{RP}^2$ with the *round metric*, obtained by quotienting S^2 (with its standard metric) by the antipodal map. Draw a fundamental domain on S^2 and its image D under stereographic projection, indicating the boundary identifications. What do geodesics on $\mathbb{RP}^2$ look like in D?

The following questions go slightly beyond the course and are for the interest of enthusiasts.

9. Given a path $\gamma : [0,1] \to \Sigma$ and a vector $V_0 \in T_{\gamma(0)}\Sigma$, show that there is a unique parallel path of vectors $V : [0,1] \to \mathbb{R}^3$ along γ satisfying $V(0) = V_0$, as follows. Split the domain into pieces $[t_{i-1}, t_i]$ such that each $\gamma([t_{i-1}, t_i])$ is contained in the image of a parametrisation, and on each of these pieces apply the modification of Picard–Lindelöf given in Q9 of Analysis and Topology Sheet 2.

10. Write down the parallel transport equations in spherical polar coordinates

$$\sigma(\theta, \varphi) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$$

on S^2. For the path $\gamma(t) = \sigma(\alpha, t)$, solve the equations explicitly to find the unique parallel path $V(t)$ with $V(0) = \sigma_\varphi$. Compute the angle between $V(0)$ and $V(2\pi)$, and compare with the intrinsic definition of Gaussian curvature.

11. Let $V \subset \mathbb{R}^2$ be the open square $(-1,1)^2$. Define two abstract Riemannian metrics on V by

$$\frac{\mathrm{d}u^2}{(1-u^2)^2} + \frac{\mathrm{d}v^2}{(1-v^2)^2} \quad \text{and} \quad \frac{\mathrm{d}u^2}{(1-v^2)^2} + \frac{\mathrm{d}v^2}{(1-u^2)^2}.$$

(a) Define a *proper ray* in V to be a smooth map $\gamma : [0,\infty) \to V$ for which the preimage of every compact set in V is compact (i.e. if $K \subset V$ is compact then $\gamma(t) \notin K$ for $t \gg 0$). Show that homeomorphisms of V take proper rays to proper rays.

(b) Prove that the surfaces equipped with the given Riemannian metrics are not isometric, but there is an area-preserving diffeomorphism between them [*Hint: for the first statement, show that exactly one of the two contains a proper ray of finite length.*]